Numerical Methods for PDE’s (ACM210)

First term: Hyperbolic Conservation Laws

• **Lecture:** Monday, Wednesday, Friday 10:00-10:55 am, 308 Firestone

• **Instructor:** Dr. Shaoqiang Tang (x5957, maotang@acm.caltech.edu, 313 Firestone)

• **Teaching Assistant:** Mr. Hannes Helgason (x4746, hannes@acm.caltech.edu, 226 Guggenheim)


• **Homework:** Assignments will be collected every two weeks. May include two types:
  
  – Project: implement schemes and perform numerical simulations (preferred language: matlab, please check with TA if you want to use other languages)
  – Problem: calculations (e.g. exact solution for a Riemann problem)

• **Grading Policies:**
  
  – Final score will be a weighted-sum of all assignments
  – For each assignment, late submission causes 10% deduction on your grade for each day after the due date
  – Discussions are encouraged, but after the discussions you should write up your solutions and codes independently

• **Supplementary Materials:** A brief will be available at the end of each week.
Tentative schedule

**Week 1:**

Part 0: General issues:
- Derivation of conservation laws
- Neglecting viscosity: parabolic to hyperbolic
- Hyperbolicity
- Examples

Part 1: Linear equations
1. Scalar equation (linear advection)

   Theoretical issues:
   - Characteristics, domain of dependence, range of influence, finite speed of propagation, smoothing discontinuous data;
   - Norms and spaces (L1, L2, BV);

Week 2:

Numerical issues:
- Truncation error, consistency, convergence, stability, Lax equivalence theorem;
- L1 stability and L2 stability;
- Schemes: upwind, Lax-Friedrichs, Lax-Wendroff;

Week 3:

Analyzing stabilities:
- Amplification factor and von Neumann stability analysis;
- Modified equation and numerical viscosity;
- CFL condition;

Computing with discontinuous initial data: dissipation and dispersion, error estimate.

2. Linear systems
• Characteristic variables (hyperbolicity): review of eigenvalue and eigen-vectors;
• Boundary conditions: boundary layer;
• Stability: energy estimate.

**Week 4:**
Part 2: Nonlinear equations
1. Scalar equation: examples
   Theoretical issues:
   • Burgers’ equation and inviscid Burgers’ equation, Cole-Hopf transform;
   • Discontinuity and weak solution;
   • Rarefaction: self-similarity solution;
   • Shock: Rankine-Hugoniot relation, entropy condition, entropy pairs; viscosity approach;
   • Riemann problem, interaction of elementary waves;
   • Special case: non-convex flux function - convex hull.

**Week 5:**
Numerical issues:
• Irreversibility: direction of time, implicit vs. explicit;
• Discrete conservations, discrete entropy;
• Convergence: total variation, Lax-Wendroff theorem;
• Godunov’s methods: entropy condition, linear system.

**Week 6:**
2. Nonlinear system:
Theoretical issues
• Examples: Euler equations
• Linearization: hyperbolicity, genuinely nonlinear, linearly degenerate, sound speed
• Elementary waves: rarefaction; shock, R-H relation and entropy condition;

• Riemann problem and Riemann solver; interaction of elementary waves.

**Week 7:**
Numerical issues
Approximate Riemann solvers: Roe’s scheme, Roe linearization for isothermal flow;

**Week 8:**
• Nonlinear stability: TVD, TVB, monotonicity preserving methods, L1 contraction method, monotone schemes;

• Monotone vs. accuracy: Godunov’s theorem, Harten-Hyman-Lax theorem;

**Week 9:**
• High resolution methods: flux limiters, slope limiters;

• ENO scheme, PPM, Central schemes.

**Week 10:**
• Method of lines;

• Front tracking: Glimm scheme;

• Multi-dimensional: difficulty and treatments, Strang splitting.
Second term: Topics for numerical methods in PDE’s

**Week 1, 2 & 3:**
Topic 1. Finite element method: an introduction
Function spaces, weak solution, variation, Galerkin method, Choice of basis functions;
Multi-scale modeling and computation of elliptic problems with oscillating coefficients.

**Week 4 & 5:**
Topic 2. Pseudo-spectral method
FFT: butterfly algorithm, complexity.
Pseudo-spectral (Fourier): aliasing and dealiasing.
Application: a nonlinear dissipative system.

**Week 6 & 7:**
Topic 3. Level set method
Moving surface, topological change; Hamilton-Jacobi equation, high-resolution methods.
Applications of level set methods to free boundary problems, e.g. multiphase flow and imagings.

**Week 8 & 9:**
Special Topics of mixed-type equations and dynamic phase transitions
Instability of Hadamard’s type, failure of entropy conditions for subsonic phase boundary, kinetic relation and nucleation criterion, Riemann solver, nonlinear stability.
Stiff source term, relaxed scheme, splitting technique and semi-continuous version, convergence and stability, multi-dimensional patterns.