In this article we have studied the nonlinear interaction between ellipticity and dissipation in a set of model equations (1.1) and established the relation between this interaction and chaos. In addition to theoretical investigations, extensive numerical simulations with these equations have been made, and different routes to chaos have been found. The numerical studies have revealed the chaotic nature of the solution.

1. Introduction

The study of chaos has been an active field since Lorenz derived his celebrated equations by mode truncation from the Rayleigh–Benard equations, a system of partial differential equations describing fluid heated from the bottom [1, 2]. Despite considerable progress in this field, it remains unclear how chaos arises from the original partial differential system. As a matter of fact, the highly nonlinear Rayleigh–Benard problem is still too complicated to be analyzed thoroughly.

To gain some understanding on that matter, a simpler set of partial differential equations was constructed as a substitute for the Rayleigh–
Benard equations for the purpose of studying chaos [3, Sect. 2.1]:

\[
\psi_t = -(\sigma - \alpha) \psi - \sigma \theta_x + \alpha \psi_{xx}, \\
\theta_t = -(1 - \beta) \theta + n \psi_x + 2 \psi \theta_x + \beta \theta_{xx},
\]

where \(\sigma, \nu, \alpha, \beta\) are positive constant coefficients, and in particular, \(\alpha < \sigma\) and \(\beta < 1\).

Although quite phenomenological, this set of equations singles out the essential mechanism of the nonlinear interaction between ellipticity and dissipation. The study is expected to yield insights into physical systems with similar mechanism, e.g., superposed fluids, Rayleigh–Benard instability, Taylor–Couette instability, and fluid flow down an inclined plane [4]. The study of such a model system may also help us, by comparison, to understand other results due to the interaction between ellipticity and dissipation, such as the phase transitions and nucleation phenomena [5].

Elemental instabilities in hydrodynamics often involve negative damping, negative dissipation, and ellipticity [4]. The Ginzburg–Landau equation and the Kuramoto–Sivashinsky equation are associated with the first two instabilities, respectively. Both equations exhibit chaotic behavior and have been explored extensively [6–13]. Ellipticity, on the contrary, is far from well understood on the aspects of chaos. The system (1.1) thus serves as a representative model for study of this important mechanism.

In this article we concern ourselves with both theoretical and numerical aspects of the spatially periodic problem of the system (1.1). Theoretically, we shall establish the existence, uniqueness, and well-posedness under the assumption that \(\theta\) keeps bounded during evolution. A linear stability analysis is also carried out to investigate the linear interaction between ellipticity and dissipation.

The nonlinear interaction between ellipticity and dissipation results in a complicated behavior of the system. We describe this by the numerical experiments in Section 3. Using the pseudo-spectral method, we study the bifurcation diagrams along two line segments over the four-dimensional parameter space. More precisely, we fix \(\alpha = 1\) and \(\beta = \alpha\). The bifurcation structures on \(\alpha = 0.15, \nu \in (0.15, 1.5)\) and \(\nu = 1, \alpha \in [0.15, 0.5]\) are displayed in Figures 1 and 2, respectively. When \(\alpha\) is fixed, the system decays to zero equilibrium for small \(\nu\). As \(\nu\) increases, “ellipticity” becomes stronger, and

![Figure 1. Computation summary on the segment \(\alpha = 0.15, \nu \in (0.15,1.5)\). C, chaos; SW, switching solution; O, zero equilibrium; P, periodic solution; QP, quasi-periodic solution; SS, steady state; TW, traveling wave.](image-url)
we find steady states, quasi-periodic solutions, traveling waves, switching solutions, and chaos. Windows of chaos and the break of periodic solutions appear after traveling waves. In Section 3.2 we present a detailed description. Meanwhile, when $\nu$ is fixed, the system decays to zero equilibrium for big $\alpha$. As $\alpha$ decreases, dissipation becomes weaker, and we find switching solutions, steady states, periodic solutions, switching solutions, chaotic solutions, and traveling waves, successively. This bifurcation structure is described in Section 3.3. For the chaotic solutions, we reveal the chaotic nature by the sensitivity to initial data, the wide band of observable frequency spectrum, the complicated Poincaré section, and ergodicity.

A related system, $\text{Eq. 1.2}$, has also been studied and has been reported elsewhere [14, 15].

2. Some theoretical results

In the following, we describe some theoretical results. Many problems still remain to be explored.

2.1. Reduction to the Lorenz equations, and the basic mechanism

In his celebrated article [1], Lorenz took the mode truncation

$$
\psi = \left( \frac{\pi^2 + a^2}{\pi a \sigma} \right) \sqrt{2} X(\tau) \sin \alpha x \sin \pi z, \tag{2.1}
$$

$$
\theta = \frac{R_c}{\pi R_a} \left[ \sqrt{2} Y(\tau) \cos \alpha x \sin \pi z - Z(\tau) \sin 2\pi z \right],
$$
to the Rayleigh–Benard problem
\[
\frac{\partial}{\partial t} \left( \nabla^2 \psi \right) = -\frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} + \nabla^4 \psi + \frac{R_s}{\sigma} \frac{\partial \theta}{\partial x},
\]
\[
\frac{\partial \theta}{\partial t} = -\frac{\partial (\psi, \theta)}{\partial (x, y)} + \frac{\partial \psi}{\partial x} + \frac{1}{\sigma} \nabla^2 \theta,
\]
(2.2)
with
\[
a^2 = \frac{\pi^2}{2}, \quad R_c = \frac{27}{4} \pi^4, \quad \tau = \frac{\pi^2 + a^2}{\sigma} - t.
\]
Retaining only terms with \(\sin ax \sin \pi z, \cos ax \sin \pi z,\) and \(\sin 2\pi z,\) he obtained
\[
\begin{align*}
\frac{dX}{dt} &= -\sigma X + \sigma Y, \\
\frac{dY}{dt} &= -Y + \nu X - XZ, \\
\frac{dZ}{dt} &= -bZ + XY,
\end{align*}
\]
(2.3)
where
\[
\nu = \frac{R_s}{R_c}, \quad b = \frac{4\pi^2}{\pi^2 + a^2}.
\]
Taking a similar truncation to Equations (1.1),
\[
\psi = \frac{1}{\sqrt{2}} X(t) \sin x,
\]
\[
\theta = \frac{1}{\sqrt{2}} Y(t) \cos x + \frac{1}{2} Z(t) \cos 2x,
\]
(2.4)
we obtain the Lorenz equations as well. In that regard our \(\nu\) and \(\sigma\) correspond to the Rayleigh number \(R_s\) and Prandtl number \(\sigma\) respectively in the Rayleigh–Benard problem.

Equations (1.1) are anticipated to exhibit complicated behavior because of their reduction to the Lorenz equations. Another interesting feature is the switching between ellipticity and hyperbolicity. To make it clear, we rewrite (1.1) as
\[
\begin{align*}
\psi_t + \sigma \theta_x &= -(\sigma - \alpha) \psi + \alpha \psi_{xx}, \\
\theta_t - \nu \psi_x - 2 \psi \theta_x &= -(1 - \beta) \theta + \beta \theta_{xx}.
\end{align*}
\]
(2.5)
The right-hand sides then contain damping and diffusion terms. If we ignore these terms temporarily, then the characteristic equation is

\[
\begin{vmatrix}
-\lambda & \sigma \\
-\nu & -2\psi - \lambda
\end{vmatrix} = 0,
\]

or

\[
\lambda = -\psi \pm \sqrt{\psi^2 - \sigma \nu}.
\] (2.6)

This means when \(\psi\) is small, the system is basically “elliptic.” Furthermore, \(\sigma\) and \(\nu\) stand for the strength of “ellipticity”; namely, the bigger these two constants, the stronger the ellipticity. The system subject to initial small disturbances are unstable due to ellipticity, and the inherent instability will cause growth of \(|\psi|\), if it overcomes the effect of the damping and diffusion terms on the right-hand side. When \(\psi^2\) surpasses \(\sigma \nu\), the system becomes basically “hyperbolic” there, and \(\psi\) ceases to grow. But the damping and diffusion would diminish the amplitude and draw the system back into elliptic regime. The instability mechanism then will operate again. The nonlinear interaction between ellipticity and dissipation may either balance at certain steady-state/traveling-wave solution or produce the back and forth switching phenomenon and complicated patterns.

Thus one may expect that bigger \(\sigma\) and \(\nu\), which correspond to stronger ellipticity, would make the solution less regular or more chaotic.

### 2.2. Existence, uniqueness, and well-posedness

Consider the spatially periodic Cauchy problem of Equations (1.1), i.e., \((t,x) \in \mathbb{R}^+ \times (0,L)\), with initial data

\[
\psi(0,x) = \psi_0(x),
\]

\[
\theta(0,x) = \theta_0(x).
\]

Here both \(\psi_0(x)\) and \(\theta_0(x)\) are \(L\)-periodic.

We denote

\[
J_k(t) = \int_0^L \left[ \frac{\partial^k \psi(t,x)}{\partial x^k} \right]^2 + \left[ \frac{\partial^k \theta(t,x)}{\partial x^k} \right]^2 \, dx.
\]

For this set of parabolic equations, we can prove that [16]:

If \(J_0(0), J_1(0), J_2(0), \) and \(J_3(0)\) are bounded, then the system admits a global unique solution \((\psi(t,x), \theta(t,x))\) if \(\psi\) or \(\theta\) keeps bounded during
evolution. Moreover, the problem is well posed in the sense that \( \forall \varepsilon > 0 \), there exists an \( \bar{\varepsilon} > 0 \), the solution \((\tilde{\psi}(t, x), \tilde{\theta}(t, x))\) to (1.1) with initial data

\[
\hat{\psi}(0, x) = \tilde{\psi}_0(x), \\
\hat{\theta}(0, x) = \tilde{\theta}_0(x)
\]
satisfies

\[
|\tilde{\psi}(t, \cdot) - \psi(t, \cdot)|_\infty, |\tilde{\theta}(t, \cdot) - \theta(t, \cdot)|_\infty < \varepsilon, \quad 0 \leq t \leq \delta_0,
\]
provided

\[
|\tilde{\psi}_0(x) - \psi_0(x)|_\infty, |\tilde{\theta}_0(x) - \theta_0(x)|_\infty < \bar{\varepsilon}.
\]

Here \( \delta_0 \) is any positive number such that one of \( \psi(t, x) \) and \( \theta(t, x) \) and one of \( \tilde{\psi}(t, x) \) and \( \tilde{\theta}(t, x) \) keeps bounded for \( t \in [0, \delta_0] \).

In fact, even in the case when the assumption of initial boundedness does not hold globally, the existence, uniqueness, and well-posedness are still true before the solution breaks up. Meanwhile, the regularity of the solution is at least as good as the initial data. The theorem ensures us that the solution to Equations (1.1) does not have any sudden blow-up; therefore, numerical simulation is possible. It may be mentioned that for (1.2), stronger theoretical results may be obtained [17].

2.3. Linear stability analysis of \((0, 0)\)

It is obvious that \((0, 0)\) is a trivial equilibrium solution. Trying a solution in the form \((\Psi e^{\lambda t - ikx}, \Theta e^{\lambda t - ikx})\) to the linearized equations, it can be found that \( \lambda \) is the root to the quadratic equation

\[
\lambda^2 + \left[\sigma - \alpha + 1 - \beta + (\alpha + \beta)k^2\right] \lambda \\
+ (\sigma - \alpha + \alpha k^2)(1 - \beta + \beta k^2) - \sigma \nu k^2 = 0;
\]

namely,

\[
\lambda_\pm = \left\{-\left[\sigma - \alpha + 1 - \beta + (\alpha + \beta)k^2\right] \mp \sqrt{\Delta}\right\}/2,
\]

where \( \Delta = [(\sigma - \alpha + \alpha k^2) - (1 - \beta + \beta k^2)]^2 + 4\sigma \nu k^2 \). The mode with wave number \( k \) is unstable if and only if

\[
\alpha \beta k^4 + \left[\alpha(1 - 2 \beta) + \sigma(\beta - \nu)\right] k^2 + (\sigma - \alpha)(1 - \beta) < 0. \quad (2.7)
\]
Close investigation of the condition shows that when \( \nu \) is small and \( \alpha/\sigma \) and \( \beta \) are greater than a certain critical value, namely, when dissipation dominates, \((0,0)\) is linearly stable. In fact, unstable modes exist if \( \nu \geq \beta + (\alpha/\sigma)(1-2\beta) \). Meanwhile, if \( \nu \) and \( \sigma \) are big or \( \alpha \) and \( \beta \) are small, namely, when ellipticity dominates, \((0,0)\) becomes unstable. More precisely, a mode is unstable if its wave number \( k \in \)

\[
\left( \sqrt{\left[ \sigma (\nu - \beta) - \alpha(1-2\beta) - \Delta_1 \right]/(2\alpha\beta)} , \right.
\]

\[
\left. \sqrt{\left[ \sigma (\nu - \beta) - \alpha(1-2\beta) + \Delta_1 \right]/(2\alpha\beta)} \right),
\]

where

\[
\Delta_1 = \sqrt{\left[ \sigma (\nu - \beta) - \alpha(1-2\beta) \right]^2 - 4\alpha\beta(1-\alpha)(1-\beta)} .
\]

If \( \alpha \) (or \( \beta \)) is zero, i.e., only one diffusion term, then all modes with wave number \( k \) big enough grow exponentially with exponent \( \lambda_+ \sim \sigma\nu/\beta \) as \( k \to \infty \). If both \( \alpha \) and \( \beta \) are zero, i.e., only damping presents, then the growth rate \( \lambda_+ \sim \sqrt{\sigma\nu} k \). For this case, any simulation is impossible, as a small error will cause immediate blow-up, a typical situation of ill-posedness of the Cauchy problem for elliptic equations.

This stability result reveals the ellipticity as the elemental instability in Equations (1.1), as well as the linear interaction between ellipticity and dissipation. Moreover, it indicates that complicated behavior may appear only for small dissipation cases and that a numerical study with vanishing viscosity of (1.1) would be very costly.

### 3. Numerical studies

We have performed numerical experiments basically with the pseudo-spectral method for calculating the space derivatives [18] and the fourth-order Runge–Kutta scheme for time integration. Most of the following numerical simulations are performed by using a sequence of refined meshes to verify convergence, although we typically show only the graphs for the mesh with 128 equidistant grid points in the spatial interval \([0,2\pi]\), i.e., \( \Delta x = 2\pi/128 \), and time step \( \Delta t = 0.005 \). Dynamical monitoring is implemented to guarantee that enough modes are included in simulating the system. The simulations are also justified by checking the numerical results with codes developed independently by two of the authors. Moreover, when applied to a slightly different system (1.2), our simulations agree with those by Deng with the finite difference scheme [19].
As there are four parameters $\alpha$, $\beta$, $\sigma$, and $\nu$ in system (1.1), we concern ourselves only with a small part of the parameter space. In particular, we take $\beta = \alpha$ and $\sigma = 1$. We report here the numerical investigations on two line segments, i.e., (i) $\alpha = 0.15$, $\nu \in (0.15, 1.5]$ and (ii) $\nu = 1$, $\alpha \in [0.15, 0.5]$. The results are summarized in Figures 1 and 2.

In this section, we first demonstrate the tools for displaying the solution by an example of quasi-periodic solution. Then we show the numerical results for parameters in the two line segments, respectively. Finally we briefly describe the drastically different behavior of the system (1.2).

3.1. Quasi-periodic solution

We first look at the case of $\alpha = \beta = 0.1$, $\sigma = \nu = 1$, and initial data

$$\psi_0 = 0.2 \sin(x - 1),$$

$$\theta_0 = 0.5 \cos x.$$

For convenience, in the following figures we use the grid numbers instead of the spatial variable $x$ as horizontal labels.

Figure 3 displays the solution $\psi(t, x)$ (solid line), $\theta(t, x)$ (dashed line), and the phase portrait $(\psi(t, x), \theta(t, x))$ at various time levels. Heavy dots are used to monitor ellipticity and hyperbolicity. A heavy dot is plotted on $\psi = 0$ at the spatial position where the system is elliptic and on $\psi = 1$ where hyperbolic.

We may make the following observations from the computation. First, at any time $t$, the spatial derivatives $\psi_x$ and $\theta_x$ are not big. About three to five humps appear. In fact, the typical wave number spectrum $|\hat{\psi}(t, k)|$ at $t = 2000$ in Figure 4a shows that very high modes contribute little in evolution. The amplitude at wave number $k = 30$ decreases to about $10^{-3}$ of that at $k = 1$, and thus the energy decreases to about $10^{-6}$ times. Almost all the energy concentrates in modes with $k \leq 15$. (See Figure 4b.) In fact, it may be found that less than $8 \times 10^{-5}$ of the total energy lies in modes with $k \geq 20$ and less than $4 \times 10^{-7}$ in $k \geq 30$. This partly explains that 128-point grid, i.e., 64-mode truncation, is good enough for simulation.

Second, the heavy dots indicate that except for the beginning (e.g., two grid points in Figure 3c), the system lies completely in the elliptic regime. This fact does not depend on the initial data. Soon we show that for different initial data, essentially the same attractor is reached. For all the simulations on system (1.1) reported in this article, we have observed that the system always lies mainly in the elliptic regime. This is quite different
Figure 3. Solution for $\alpha = \beta = 0.1, \sigma = \nu = 1$: (a) $t = 0$; (c) $t = 20$; (e) $t = 1000$; (g) $t = 1020$; (b, d, f, h) Phase portraits corresponding to (a, c, e, g), respectively.
Figure 4. Wavenumber spectrum: (a) $|\hat{\phi}(2000, k)|$; (b) energy percentage bar.
from the case of Equations (1.2), where hyperbolicity plays a more important role. (See Section 3.4.)

Third, from this figure we find hardly any clue on how the solution evolves. But by stacking the phase-plane curves \( (\psi(t,x), \theta(t,x)) \) at time \( t = t_0 + k\delta \), for \( t_0 = 1000 \), \( k = 0, 1, \ldots, 50 \), and \( \delta = 20 \), we obtain an interesting envelope in Figure 5. We obtain the same envelope for different \( t_0 \) and \( \delta \), provided that \( t_0 \) is sufficiently large. This envelope persists under small perturbation. For instance, we perturb the solution at \( (t,x) = (2000, 99*2\pi/128) \) by 1% and continue the computation. The solutions for the unperturbed problem and the perturbed problem are displayed in Figure 6. They turn out to be indistinguishable. In Figure 6d, \( \psi^{(1)}(t,99\Delta x) \) coincides with \( \psi^{(2)}(t,99\Delta x) \) as well.

The envelope shown here can be viewed as a projection of the attractor in the context of partial differential equations. The attractor is an asymptotic state that is insensitive to initial data. We have tried a variety of initial data, and the solutions converge essentially to the same attractor. Some such initial data and corresponding time series envelopes are shown in Figure 7. Due to the invariance of the system (3.3.1)

\[
\{ \psi \rightarrow -\psi, \theta \rightarrow \theta, x \rightarrow -x, t \rightarrow t \},
\]
a \( \psi \)-reflected envelope is also reached for certain initial data.

We may better capture the features of the complicated behavior by applying some other tools. One of them is the time series, taken as \( \psi(t,x_0) \) and \( \theta(t,x_0) \) with a fixed spatial position \( x_0 \).

![Figure 5. Phase-portrait envelope for \( t \in [1000,2000] \).](image)
Figure 6. Persistence under perturbation: (a) $\psi(0, x)$; (b) $\psi(100, x)$; (c) $\psi(200, x)$; (d) $\psi(t, x_0)$.

Figure 8 shows the time series for $x_0 = \frac{99}{125} \times 2\pi$. (In all the following cases, we take this value for $x_0$.) It suggests that the evolution can be separated into two parts. In the first 450 time units or so, the series are irregular. This is the transient part. The phase-plane portrait is irregular (Figure 9a). Afterward, the solution looks well evolved, and a regular pattern forms. It is significant that the corresponding phase portrait in Figure 9b reproduces the attractor (Figure 5), where the solution in the whole spatial interval has been plotted. We refer to this plot as a time series envelope later on. At any other spatial position, we obtain the same plot. Viewing a point in the $(\psi, \theta)$-plane as a state, each state of the attractor is reached at any particular spatial position. This phenomenon is the ergodicity [20].

Figure 10 displays the frequency spectrum (the Fourier transform) of the time series $\psi(t, x_0)$ for $t \in [1000, 2000]$. It consists of a wide band of observable frequencies. Actually, the peaks here are just at frequencies that are combinatorial sums of two fundamental ones. These two frequencies are better recognized in Figure 10b, which is an enlarged portion. One is about $f_c \approx 0.112$, corresponding to the humps of the time series ("carrier"). The other is about $f_e \approx 0.011$, corresponding to the envelope of the humps. These two frequencies are not commensurate, resulting in the quasi-periodicity of the solution. The torus, resulting from the quasi-periodicity of these two frequencies, underlies the features of the phase-portrait envelope in Figure 5 and the time-series envelope in Figure 9b.
Figure 7. Attractor under different initial data: (a, c, e, g) Initial data; (b, d, f, h) corresponding attractor.
Figure 8. Time series: (a) $\psi(t, x_0)$; (b) $\theta(t, x_0)$.

Figure 9. Phase-plane plot of the time series: (a) $t \in [0, 200]$; (b) $t \in [1000, 2000]$. 
Besides time series, it is also helpful to view the solution from another perspective. Consider the truncated Fourier series

$$\psi(t, x) = \frac{2}{\sqrt{N}} \sum_{n=0}^{N/2} (a_n(t) \cos nx + b_n(t) \sin nx),$$

$$\theta(t, x) = \frac{2}{\sqrt{N}} \sum_{n=0}^{N/2} (c_n(t) \cos nx + d_n(t) \sin nx).$$

It tends to the exact solution when $N \to \infty$. This truncated system is of dimension $2 \times (2N + 1)$; namely, there are $2 \times (2N + 1)$ variables: $a_n$’s, $b_n$’s, $c_n$’s, and $d_n$’s. A useful tool is the Poincaré section, where the point on the trajectory is recorded as the trajectory cuts a fixed hyperplane transversally. Typically, a steady-state solution will result in no point, a time-periodic solution a finite number of points, a quasi-periodic solution a finite number of closed curves, and a chaotic solution a dense point set.
In Figure 11, the Poincaré section records \((a_1(t), a_2(t))\) of the trajectory when it passes through the plane \(b_1(t) = 0\) with \(db_1(t)/dt \neq 0\). Some enlarged details are put in the other subplots. The section contains two closed curves, thus indicating the quasi-periodicity of the solution.

The time series, its spectrum, and the Poincaré section, as we have observed, seem to capture the features of the solution more effectively. In the following sections, we use them mainly for displaying the solutions.

### 3.2. Bifurcation on \(\alpha = 0.15, \nu \in (0.15, 1.5]\)

The basic mechanism of the system (1.1) is the interaction between ellipticity and dissipation. In this section, we investigate the system by changing \(\nu\) while fixing \(\alpha = 0.15\) and \(\sigma = 1\). As mentioned before, \(\nu\) stands for the strength of ellipticity. For the segment \(\nu \in (0.15, 1.5]\), the numerical simulations have been made for \(\nu = 0.15 + 0.01k\), for \(k = 1, \ldots, 135\). Each bifurcation value is detected by successive bisection on the interval where the bifurcation occurs, up to the accuracy of 0.001. Therefore, the bifurcation structure finer than this scale may be beyond the scope of our study reported here.

We know from the previous example that the asymptotic behavior of the solution does not depend sensitively on initial condition. For definiteness, all subsequent computations employ the same initial data

\[
\psi_0 = 0.2 \sin(x - 1),
\]

\[
\theta_0 = 0.5 \cos x.
\]

![Figure 11. Poincaré section: (a) The whole section; (b, c, d) enlarged section (part).](image)
1. \( \nu \in (0.15, 0.526) \), zero equilibrium: When \( \nu \) is small, dissipation terms dominate the evolution and drive the system to zero equilibrium. The decaying of the solution is well described by the linear analysis. Manipulation on (2.7) predicts that the bifurcation value is \( \nu = 0.525625 \), when the mode with \( k = 2 \) becomes linearly unstable. The solution for \( \alpha = 0.52 \) is plotted in Figure 12.

2. \( \nu \in [0.526, 0.615] \), steady-state solution: The system maintains a steady wave pattern when \( \nu \) increases to the critical value. The dependence of the wave amplitude of \( \psi \) on \( \nu \) is listed below. Bigger \( \nu \) is related to bigger amplitude, indicating the stronger ellipticity. For \( \nu = 0.56 \), see Figure 13. It is spatially a \( \pi \)-periodic pattern. However, when \( \nu \) is 0.61, the steady state is a two-hump one, but not \( \pi \)-periodic (Figure 14).

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>0.53</th>
<th>0.54</th>
<th>0.55</th>
<th>0.56</th>
<th>0.57</th>
<th>0.58</th>
<th>0.59</th>
<th>0.60</th>
<th>0.61</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude</td>
<td>0.0523</td>
<td>0.0937</td>
<td>0.1203</td>
<td>0.1407</td>
<td>0.1569</td>
<td>0.1700</td>
<td>0.1805</td>
<td>0.2075</td>
<td>0.2418</td>
</tr>
</tbody>
</table>

3. \( \nu \in [0.615, 0.643] \), quasi-periodic solution: For this range of \( \nu \), a quasi-periodic solution appears. In fact, the solution seems to represent a modulation of a wave traveling slowly from right to the left. For instance, when \( \nu = 0.63 \), the time series \( \psi(t, x_0) \) behaves almost like a wave. See Figure 15. The two frequencies are \( f_c = 0.037 \) and \( f_c = 0.0025 \).

Figure 12. Zero equilibrium for \( \nu = 0.52 \): (a) \( \psi(t, x) \); (b) \( |\psi(t, x_0)| \).
4. $\nu \in [0.643, 0.698)$, steady-state solution: The system returns to its steady state as $\nu$ increases further. However, instead of a two-hump solution, there are three humps (Figure 16). In fact, as the ellipticity gets stronger, more modes become linearly unstable.
5. $\nu \in [0.698, 0.739)$, traveling-wave solution: The traveling-wave solution in this region of $\nu$ has two humps. See Figure 17.

6. $\nu \in [0.739, 0.878)$, chaos: Bifurcation at $\nu = 0.739$ causes the appearance of chaos. We demonstrate the solution for $\nu = 0.8$ by the time series, its wide observable spectrum, and the Poincaré section in Figure 18.
7. $\nu \in [0.878, 0.885)$, switching solution: For $\nu$ in this small range, the system asymptotically switches between two four-hump steady states. See Figure 19. We have a more comprehensive description of this kind of switching solution in Section 3.3.

8. $\nu \in [0.885, 1.076)$, traveling-wave solution: When $\nu = 0.885$, the system tends to traveling-wave solution. The motion is clearly revealed by
comparing $\psi(t, x)$ at successive time levels $t = 1000, 1010, 1020,$ and 1030 in Figure 20 ($\nu = 1$). In fact, when $\nu$ increases, we note that both the time-series frequency and the wave speed increases. See the table below.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>0.89</th>
<th>0.90</th>
<th>0.91</th>
<th>0.92</th>
<th>0.93</th>
<th>0.94</th>
<th>0.95</th>
<th>0.96</th>
<th>0.97</th>
</tr>
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<tbody>
<tr>
<td>Frequency</td>
<td>0.070</td>
<td>0.072</td>
<td>0.073</td>
<td>0.075</td>
<td>0.076</td>
<td>0.078</td>
<td>0.079</td>
<td>0.081</td>
<td>0.082</td>
</tr>
<tr>
<td>Wave speed</td>
<td>0.2199</td>
<td>0.2262</td>
<td>0.2293</td>
<td>0.2356</td>
<td>0.2388</td>
<td>0.2450</td>
<td>0.2482</td>
<td>0.2545</td>
<td>0.2576</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>0.98</th>
<th>0.99</th>
<th>1.00</th>
<th>1.01</th>
<th>1.02</th>
<th>1.03</th>
<th>1.04</th>
<th>1.05</th>
<th>1.06</th>
</tr>
</thead>
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<tr>
<td>Frequency</td>
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<td>0.091</td>
<td>0.092</td>
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<tr>
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<td>0.2670</td>
<td>0.2733</td>
<td>0.2765</td>
<td>0.2796</td>
<td>0.2827</td>
<td>0.2859</td>
<td>0.2922</td>
<td>0.2985</td>
</tr>
</tbody>
</table>

9. $\nu \in [1.076, 1.099]$, quasi-periodic solution: The solution becomes quasi-periodic for $\nu$ in this interval. Figure 21 shows the time series and the corresponding spectrum for $\nu = 1.08$. The carrier frequency is $f_c = 0.096$, and the envelope frequency is $f_e = 0.005$.

10. $\nu \in [1.099, 1.280]$, steady-state solution: Steady-state solution appears at $\nu = 1.099$. In this case, the time series $\psi(t, x)$ tends to a fixed value. In the space of $(a_n, b_n, c_n, d_n)$, it is related to a fixed point, and the Poincaré section is an empty set.

11. $\nu \in [1.280, 1.3105]$, periodic solution: Increasing $\nu$ to 1.28, we find periodic solutions, which correspond to a close orbit in $(a_n, b_n, c_n, d_n)$-space formed after a Hopf bifurcation.

12. $\nu = 1.31055$, break of the periodic solution, period-doubling bifurcation: The periodic solution (close orbit) breaks up after a long run, and the solution moves up to another periodic state.

13. $\nu \in (1.3106, 1.5]$, chaos: For $\nu$ even bigger, the system is chaotic.

In this section, we have investigated the formation of complicated patterns as increasing $\nu$, the strength of ellipticity. The chaotic solution appears after the break of periodic solution, i.e., close orbit in the wave space. More details can be found in [21].

### 3.3. Bifurcation on $\nu = 1$, $\alpha \in [0.15, 0.5]$.

As well as varying $\nu$, varying $\alpha$ also affects the interaction between ellipticity and diffusion, thereby changing the asymptotic behavior of the solution. We investigate system (1.1) under different $\alpha$, while fixing $\nu = 1$.

Again we take $x_0 = \frac{99}{128} \times 2\pi$ for time series and initial data,

$$\psi_0 = 0.2\sin(x - 1),$$

$$\theta_0 = 0.5\cos x.$$ 

We have performed numerical experiments for $\alpha = 0.15 + 0.01k$, where $k = 0, \ldots, 35$, and detected the bifurcation value by bisection up to the accuracy of 0.001.
Figure 18. For $\nu = 0.8$: (a) $\psi(t, x_0)$; (b) spectrum of $\psi(t, x_0)$; (c) Poincaré section at $c_z(t) = 0$ with $dc_z(t)/dt \neq 0$.

1. $\alpha \in (1/3, 0.5]$, zero equilibrium: As indicated from the stability analysis, the solution decays gradually to zero equilibrium for big dissipation. More precisely, as the spatial period is fixed to be $2\pi$, the growth exponent $\lambda_+ = 0$, $-2$ for the fundamental mode ($k = 1$). The dissipation tends to cause the decaying of the amplitude, while the nonlinearity promotes the exchange
of energy among different modes. It can be found that the critical value of $\alpha$ is $\frac{1}{\nu}$, when the mode with $k = 2$ starts to be unstable. In Figure 22, we demonstrate the decaying by the three-dimensional plot of $\psi(t, x)$ and the time series $\psi(t, x_0)$ at $\alpha = 0.4$. It has been observed that the bigger $\alpha$, the faster the decaying. This is also implied in the linear stability results.

Figure 19. Switching between steady states for $\nu = 0.88$: (a) $\psi(t, x)$; (b) $\psi(t, x_0)$.

Figure 20. Traveling wave solution for $\nu = 1$: $\psi$ at $t = 1000, 1010, 1020, 1030$. 
Figure 21. Quasi-periodic solution for $\nu = 1.08$. (a) Time series $\psi(t, x_0)$; (b) frequency spectrum of $\psi(t, x_0)$.

Figure 22. Zero equilibrium for $\alpha = 0.4$: (a) $\psi(t, x)$; (b) time series $\psi(t, x_0)$. 
2. $\alpha \in [0.306, 1/3]$, switching solution: When dissipation decreases to $\alpha = \frac{1}{3}$, a nonzero wave pattern forms. The system switches between two steady states. The solution is displayed in Figure 23. For one steady state, there are two humps, intersecting with $\psi(t, x) = 0$ at four consequent points $x_i, i = 1, 2, 3, 4$. At $x_1$ and $x_3$, $\psi_1 > 0$; at $x_2$ and $x_4$, $\psi_2 < 0$. From the snapshots in Figure 24, we may observe that during the switching, $\psi(t, x)$ keeps unchanged at $x_2$ and $x_4$. After the switching, the system reaches the other steady state, for which $\psi_2 < 0$ at $x_1, x_3$, and $\psi_2 > 0$ for $x_2, x_4$. In the next switching, $\psi(t, x)$ keeps unchanged at $x_1, x_3$, and the sign of $\psi_2$ changes again. The fixation at points, where $\psi_2 < 0$, can be explained by a crude local analysis of Equation (1.1). More features about the switching solution are described in a similar case of $\alpha = 0.22$ below.

3. $\alpha \in [0.266, 0.306]$, steady-state solution: The system converges to a steady-state solution. See Figure 25.

4. $\alpha \in [0.251, 0.266]$, periodic solution: We find a time-periodic pattern for this range of $\alpha$. There are two humps in $\psi(t, x)$. Like in the previous switching solution case, $\psi(t, x)$ remains unchanged at the two points where $\psi = 0$ and $\psi_2 < 0$. The frequency can be found as $f = 0.051$. See Figure 26.

5. $\alpha \in [0.217, 0.251]$, switching solution: In this case the solution is again observed to switch between two steady states. More precisely, a steady state forms, yet breaks after a long run. Then another steady state forms, which
also is not really stable, and falls back to the original one again. For example, the solution for $\alpha = 0.24$ is depicted in Figure 27 and the time series in Figure 28. The time-series phase portrait in Figure 29 also describes the break. The steady states are represented by two saddle points here. Each switching corresponds to the trajectory from the unstable manifold of one saddle point to the stable manifold of the other saddle point.

Moreover, it is interesting that these two steady states differ from each other only by a phase shift of $\pi/2$. Figure 30 shows the steady states and the discrepancy between them up to the phase shift.

6. $\alpha \in [0.192, 0.217]$, chaos: Below $\alpha = 0.217$, steady states are difficult to be maintained, and the solution switches more often. See Figure 31 for $\alpha = 0.217$ and $\alpha = 0.216$. Further decreasing $\alpha$, the switching pattern is destroyed and the system becomes chaotic.

Figure 24. Switching between steady states for $\alpha = 0.32$: (a) One switching $\psi(t,x)$; (b) another switching $\psi(t,x)$.

Figure 25. Steady-state solution for $\alpha = 0.28$: $\psi(t,x)$. 
Figure 26. Periodic solution for $\alpha = 0.26$: (a) $\psi(t, x_0)$; (b) $\psi(t, x)$ for $t = 1000, 1030, \ldots, 1600$.

Figure 27. Break of steady state for $\alpha = 0.24$: (a) Evolution of $\psi$; (b) evolution of $\theta$. 
The chaotic nature of the solution for $\alpha = 0.2$ is revealed by the time series in Figure 32, the frequency spectrum in Figure 33, and the Poincaré section in Figure 34.

Within this interval of $\alpha$, a finer bifurcation structure may exist. As a matter of fact, a steady-state solution is found for $\alpha = 0.2043$, whereas for both $\alpha = 0.2042$ and $\alpha = 0.2044$, time-periodic solutions are found.

7. $\alpha \in [0.15, 0.192]$, traveling-wave solution: We find traveling waves for this range of dissipation coefficients.
In this section, a complicated pattern forms as the dissipation coefficient $\alpha$ decreases. Chaos has been observed after the break of the switching solution.

3.4. The conservative form system (1.2)

Besides (1.1), we have also studied system (1.2). They differ from each other only by the nonlinear term, namely $(\psi \theta)$ versus $2\psi \theta$. We thus call (1.2) the conservative form and (1.1) the nonconservative form. Most theoretical results in the previous section are true for (1.2) as well. However, numerical experiments discover drastic differences between these two forms. For more details, please refer to [15, 16, 19, 22].
When the dissipation is strong, the system again decays steadily to zero equilibrium. As the dissipation decreases, so far as our numerical experiments are concerned, no complicated pattern is observed. The solution always approaches steady state. The $\theta$-plot actually becomes a set of discrete spikes. For instance, Figure 35 displays the solution for $\alpha = \beta = 0.1$ and $\sigma = \nu = 1$, with initial data

$$\psi_0 = 0.5 \sin x,$$

$$\theta_0 = -1 + 0.5 \cos x.$$
Figure 34. Chaos for $\alpha = 0.2$: Poincaré section at $c_2(t) = 0$ with $dc_2(t)/dt \neq 0$.

Figure 35. Solution for the conservative form with $\alpha = 0.1$: (a) $t = 0$; (b) $t = 2$; (c) $t = 5$; (d) $t = 8$; (e) $t = 9$; (f) $t = 20$; (g) $t = 40$; (h) $|\hat{\phi}(40, k)|$. 
The steady-state solution consists of four humps with equal amplitude. The evolution consists of three parts, namely the sharpening of the wave profile and generation of more humps, the coalescence of some humps, and the balancing among humps. Compared with the nonconservative case, it may be observed that the hyperbolicity plays a more important role here. The steady state is a nonlinear balance among ellipticity, hyperbolicity, and dissipation.

It has been found that the smaller the dissipation, the greater the number of humps and the bigger the amplitude of the wave.

Since large gradients occur in the evolution of (1.2) (Figure 35h), more grid points and smaller time steps are required. As $\alpha$ decreases, the computing time increases dramatically.

### 4. Discussion

We have made a comprehensive attempt to establish a link between the nonlinear interaction between ellipticity and dissipation, and the generation of chaos. With the model equation (1.1), which can be said to be a simplification of the Rayleigh–Benard equations, we have demonstrated numerically that counterbalanced by dissipation, ellipticity may lead to chaotic behavior of the solution.

Theoretically, we have established the existence, uniqueness, and well-posedness of the solutions under certain conditions. The analysis of linear stability reveals that complicated behavior exists only in cases of small dissipation. Our theoretical results thus legitimize our numerical computations, as well as clarify some phenomena in the simulations.

Many questions in connection with these systems remain unanswered. For instance, we have studied a slightly modified version, i.e., the conservative form (1.2). Numerical experiments have discovered a drastically different behavior. For the conservative form with small dissipation, the solution tends to steady state, represented by a set of discrete spikes. The decrease of dissipation results in the increase in the number of spikes and their steepening. It is indeed strange that such drastic differences can result from the seemingly small difference in one nonlinear term.

We have performed numerical simulations for the cases (i) varying $\nu \in (0.15, 1.5]$ with fixed $\alpha = 0.15$ and (ii) varying $\alpha \in [0.15, 0.5]$ with fixed $\nu = 1$. From the numerical experiments, we have discovered different routes to chaos, including the break of the time-periodic solution and the break of the switching solution. The chaotic nature of the solution has been clearly shown by the sensitivity to initial data, the wide band of observable frequency spectrum, the complicated Poincaré section, and the ergodicity. We also observed that basically the stronger ellipticity and smaller dissipation are related to bigger amplitude of the wave, and in case of traveling waves,
bigger wave speed. It is conceivable that increasing $\nu$ is comparable with decreasing $\alpha$. With these numerical studies, we have gained more understanding of chaos from the perspective of partial differential equations, as well as the interaction between ellipticity and dissipation.

There are still many interesting problems left with these equations. Since there are many coefficients in (1.1), the system is expected to contain other striking phenomena. Thus a full bifurcation diagram is needed for complete understanding. The bifurcation region, instead of bifurcation interval, needs to be mapped. We have made some computations for different $\nu$, $\alpha$ along the line $\nu = 0.53 + 0.94(\alpha - 0.5)/3$. The system converges to steady-state solutions with two humps. This suggests that there is a connected parameter region for two-hump steady-state solutions, containing this line, as well as the two bifurcation intervals discovered in the previous simulations, i.e., $\nu \in [0.526, 0.615]$, $\alpha = 0.15$, and $\alpha \in [0.266, 0.306]$.

In the case of ordinary differential equations, there is some powerful software, such as AUTO and DSTOOL, for detecting the bifurcation structure. Those tools are not applicable for our system, being a set of partial differential equations. The complicated, and possibly very fine, bifurcation structure is challenging and attractive. Moreover, when the dissipation becomes small enough, hyperbolicity may play a more important role. The interaction among the ellipticity, hyperbolicity, and dissipation will certainly lead to more complex patterns in the solution, which also pose a big challenge to the numerical simulation.

To further the understanding of the chaotic nature of the solutions, it may be desirable to proceed to determine the Lyapunov exponent and the fractal dimension of the attractor reconstructed from the time series and also to study the “basin” of the attractors, as numerical experiments also show that there might be multiple attractors under the same parameters.

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