Analysis of Multi-transmitting Formula for Absorbing Boundary Conditions

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Abstract: In this paper, we analyze the multi-transmitting formula (MTF) proposed by Liao\textit{et al} [19]. From the computed reflection coefficients for the fully discrete MTF boundary conditions, we suggest choices for the artificial wave propagation speed, which are different from Liao’s original choice. Theoretical and numerical studies for various incidence angles demonstrate that the suggested choices effectively reduce spurious reflections.

Keywords:
Absorbing boundary condition, multi-transmitting formula, reflection coefficient, artificial wave propagation speed.

1 Introduction

Artificial boundaries are commonly introduced for simulating wave propagations in unbounded domains, or for bridging different spatial scales in materials science and engineering [22, 24]. Special numerical treatments are needed on these boundaries to close the computing procedure. Using an absorbing boundary condition, one mimics the transparent wave propagation, in order to avoid spurious reflections. Systematic explorations on absorbing boundary conditions started from the pioneering work by Engquist, Clayton and Majda [3, 5, 6], as well as that by Reynolds [25].

Depending on its formulation, a boundary condition may be either local or non-local. Exact solution based approaches usually lead to nonlocal conditions, such as the time history treatment [2, 4], the Dirichlet-to-Neumann map [9, 16], and the difference potential method [26, 32], etc. While high orders of accuracy are attained, these methods usually demand considerable computing resource and complicated implementation.

In order to reduce numerical costs, local absorbing boundary conditions have been developed. Many of them are constructed on the basis of one-way wave equations, which characterize wave propagation toward certain outgoing directions [7, 23, 30]. This type of local conditions include, to name a few, the Higdon-Keys condition, and the velocity interfacial conditions [12, 13, 14, 15, 28]. Theoretically speaking, one may improve the order of accuracy as much as needed, by introducing increasingly high order spatial and temporal derivatives. In addition, an incident wave is typically not unidirectional in applications. By using the product of one-way wave operators, perfect absorption may be reached at certain directions, and

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the spurious reflections are reduced at nearby directions. The operator product also contains high order derivatives. A finite difference implementation of higher order derivatives makes use of more grid points, and the resulted algorithm becomes more complicated and more nonlocal. Therefore, simplicity is of great importance to make a boundary condition practically feasible. A nice example is the space-time extrapolation method [30].

It is worth mentioning that locality may be improved by introducing auxiliary variables [10, 11]. We also remark that one may alternatively choose a thin layer as an artificial boundary, and take either a perfect matched layer method [1], or a control boundary condition [17]. For more complete surveys on absorbing boundary conditions, please refer to [7, 8, 10, 11, 23, 27, 31, 33] and references therein.

In this paper, we are concerned with a class of multi-transmitting formula (MTF) boundary conditions, developed mainly in the geophysics community by Liao et al. [18, 19, 20, 21]. Closely related to Higdon’s space-time extrapolation method, MTF bears a nice feature of simplicity and clarity, which is of particular interest for high-order boundary treatments. As a matter of fact, an MTF boundary condition involves only two parameters, namely, an artificial wave propagation speed and a space time mesh ratio. With this simple parameter setting, the main focus of this study is the choice of the artificial wave propagation speed, in order to suppress spurious reflections most effectively. Through careful analysis, we propose new choices for this parameter. Theoretical and numerical results verify the effectiveness for the choice.

The rest of this paper is arranged as follows. In Section 2, we present a clear and rigorous formulation for MTF in two steps. First, we derive a semi-continuous form of boundary condition along a straight line emanating from the boundary with a slope of the selected artificial wave propagation speed. Then, the fully discrete boundary conditions are obtained through interpolations with the nearest neighboring grid points of the interior solver. In Section 3, we analyze several MTF boundary conditions in their fully discrete forms. We notice that Liao only performed analysis on the semi-continuous forms [21]. By Taylor expansions, we obtain the continuous forms for various MTF boundary conditions. This facilitates calculating the reflection coefficients for monochromatic waves at different incidence angles. We then suggest choices for the artificial wave propagation speed to reduce reflections for a wide range of incidence angles. Interestingly, our choices differ from those suggested by Liao. Moreover, Higdon’s space-time extrapolation turns out to be a special case, while the MTF takes a much simpler form than Higdon’s formulation in general. Taking a standard centered-time-centered-space scheme as the interior solver, we compute theoretically and numerically the reflection coefficients. The results verify the effectiveness for the proposed artificial wave propagation speeds. Numerical tests are reported in Section 4, including a time harmonic wave test and a Gaussian source study. We make some concluding remarks in Section 5.
2 Formulation of the multi-transmitting formula

Consider the Cauchy problem for the wave equation in two space dimensions as follows.

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad (x, y) \in \mathbb{R}^2. \tag{1}$$

In practical simulations, one takes a finite computing domain. Under the natural assumption that all waves propagate outward, appropriate boundary conditions are needed on the artificial boundary of the selected computing domain. The design of such boundary conditions is the central topic for computing wave propagations in infinite domains, as well as for multiscale computations.

To start with, we consider a numerical boundary $x = 0$ in Fig. 1. A plane wave at an incidence angle $\theta \in (-\pi/2, \pi/2)$ takes the form of $u(x, y, t) = f(x \cos \theta + y \sin \theta - ct)$. Projecting to the $x$–axis, we find that it solves the wave equation in one space dimension with an apparent propagation speed $c_a(\theta) = c/\cos \theta$ [21].

$$u_{tt} = c^2_a(\theta)u_{xx}. \tag{2}$$

If the incidence angle $\theta$ is known, one decomposes the wave equation (2), and design an exact absorbing boundary condition [12, 13, 14, 15]. However, this is not the case in real applications such as seismology or electro-magnetics. A generic incident wave contains many plane wave components, with incidence angles in a certain range. Though the exact boundary conditions do not apply, one may guess an artificial wave propagation speed $a$, and design a boundary condition that absorbs spurious reflection most effectively around $\theta = \arccos c/a$. This is the key idea in multi-transmitting formula (MTF) first proposed by Liao et al [18, 19, 21].

MTF boundary conditions are constructed in two steps.
First, we make a semi-discretization for the time dimension with a step size $\Delta t$. Along a straight line emanating from a boundary point $(X_0, t^{n+1})$ with slope $-a$ in the physical plane, the Newton’s formula holds for any $C^N$ function $f(\xi)$ that

$$f(\xi) = \sum_{\beta=1}^{N} (-1)^{\beta+1} C_N^\beta f(\xi - \beta \Delta \xi) + f^{(N)}(\xi^*)(\Delta \xi)^N, \quad \xi^* \in (\xi - N \Delta \xi, \xi). \quad (3)$$

Here $\xi = x + at$, $C_N^\beta = \frac{N!}{\beta!(N-\beta)!}$, and $\Delta \xi = \sqrt{1 + a^2 \Delta t}$. See Fig. 2 subplot (a).

Applying this to the wave propagation problem and ignoring the residual, we obtain an $N$-th order multi-transmitting formula (MTF-N for short) boundary condition in a semi-continuous form [18, 19, 21].

$$u_0^{n+1} = \sum_{\beta=1}^{N} (-1)^{\beta+1} C_N^\beta u_{\beta}^{n+1-\beta}.$$ \quad (4)

Here $u_{\beta} = u(X_{\beta}, t^n)$ at the point $X_{\beta} = X_0 + \beta(a \Delta t)$. We remark that (4) is a general numerical boundary condition, regardless of the original partial differential equation. Interested readers are referred to [21, 27] for alternative derivations.

In the second step, we perform a linear interpolation to reconstruct $u_{\beta}^{n+1-\beta}$ with the two nearest neighboring points.

$$u_{\beta}^{n+1-\beta} \approx (1 - \gamma_{\beta}) u_{J_{\beta}}^{n+1-\beta} + \gamma_{\beta} u_{J_{\beta}+1}^{n+1-\beta}.$$ \quad (5)

Here the space and time mesh ratio $\eta = \Delta x/\Delta t$, and the integer and fractional parts of $a/\eta$ are denoted as $J = [a/\eta]$ and $\gamma = \{a/\eta\}$, those of $\beta a/\eta$ as $J_{\beta} = [\beta a/\eta]$ and $\gamma_{\beta} = \{\beta a/\eta\}$. See Fig. 2 subplot (b).

An MTF-N boundary condition then follows.

$$u_0^{n+1} = \sum_{\beta=1}^{N} (-1)^{\beta+1} C_N^\beta (1 - \gamma_{\beta}) u_{J_{\beta}}^{n+1-\beta} + \gamma_{\beta} u_{J_{\beta}+1}^{n+1-\beta}. \quad (6)$$

We make several remarks. First, the Higdon’s $N$-th order space-time extrapolation falls as a special case with $a = \eta$ [12].
Secondly, higher order upwind interpolations may also be adopted. For example, with a quadratic upwind interpolation, an MTF-N with quadratic interpolation (MTF-NQ for short) takes the following form
\[ u_{n+1}^0 = \sum_{\beta=1}^{N} (-1)^{\beta+1} C_\beta^1 \frac{1}{2} (1-\gamma_\beta)(2-\gamma_\beta)u_{j_\beta}^n + \gamma_\beta(2-\gamma_\beta)u_{j_\beta+1}^n - \frac{1}{2} \gamma_\beta(1-\gamma_\beta)u_{j_\beta+2}^n. \] (7)

Finally, for wave propagations with an apparent propagation speed \( c_a \) toward the left boundary, it holds that
\[ \frac{\partial u}{\partial t} = -c_a \frac{\partial u}{\partial x}. \] (8)

Therefore, the residual \( f^{(N)}(\xi^*)(\Delta \xi)^N \) is on the order \( O((c_a - a)\sqrt{1 + a^2\Delta t})^N) \) [21]. This provides a local error estimate. In particular, the differential form for (4) is
\[ \left( \frac{\partial}{\partial t} - a \frac{\partial}{\partial x} \right)^N u = 0. \] (9)

3 Reflection coefficients for MTF-N

In this section, we explore the properties of MTF boundary conditions through reflection coefficients. First, we make Taylor expansions for MTF’s and find their continuous forms. Secondly, we investigate the reflection coefficient for plane waves with different incidence angles. This allows us to optimize the choice of \( a \). Finally, we present a reflection coefficient study with a standard centered-time-centered-space (CTCS) scheme as the interior solver.

3.1 Reflection coefficient

A reflection coefficient \( R \) is the standard tool for analyzing wave phenomena. It may be introduced in a plane wave with an incidence angle \( \theta \in [0, \pi/2) \) at the left end.
\[ u = e^{ik(\xi \cos \theta + y \sin \theta)} + R e^{ik(\xi \cos \theta - y \sin \theta)}. \] (10)

The reflection coefficient \( R \) represents the spurious reflection caused by an artificial boundary condition. For a boundary condition in the continuous form
\[ \left[ \prod_{j=1}^{N} \left( \frac{\partial}{\partial t} - a_j \frac{\partial}{\partial x} \right) \right] u = 0, \] (11)
where \( a_j \)’s are parameters, it holds that [12]
\[ R(\theta) = - \prod_{j=1}^{N} \frac{c - a_j \cos \theta}{c + a_j \cos \theta} = - \prod_{j=1}^{N} \frac{c_a(\theta) - a_j}{c_a(\theta) + a_j}. \] (12)
3.2 MTF-1

The MTF-1 boundary condition reads

\[ u_{n+1}^0 = (1 - \gamma)u_J^n + \gamma u_{J+1}^n. \]  

(13)

We remark that this upwind-scheme like formula serves as a boundary condition instead of an interior solver. Moreover, it uses the artificial wave propagation speed \( a \), not the true physical propagation speed \( c \) or the apparent propagation speed \( c_a \).

Taking the Taylor expansions and recall that \( J + \gamma = a/\eta \), we obtain the following continuous form.

\[ \left( \frac{\partial}{\partial t} - a \frac{\partial}{\partial x} \right) u = O(\Delta x). \]  

(14)

The reflection coefficient is

\[ R(\theta) = \frac{c - a \cos \theta}{c + a \cos \theta}. \]  

(15)

Due to the importance of the normal incidence wave, we suggest to take \( a = c \). With this choice, the normal incidence wave (\( \theta = 0 \)) propagates transparently at the artificial boundary. In the continuous form, this is the first order boundary condition proposed by Engquist and Majda [5, 6].

Noticing the Courant-Friedrichs-Lewy stability condition \( c < \eta \), we have \( J = 0 \), and \( a = \gamma \eta \). The MTF-1 then becomes

\[ u_{0}^{n+1} = (1 - a/\eta)u_{0}^n + (a/\eta)u_{1}^n. \]  

(16)

We remark that if \( c > \eta \), one needs more points at the previous time layer or implicit boundary conditions to maintain the stability [13].

3.3 MTF-2

The MTF-2 involves two time layers. Depending on the value of \( \gamma \), the straight line emanating from \((x_0, t^{n+1})\) with slope \(-a\) intersects the time layer \( t^{n-1} \) either between \( x_{2J} \) and \( x_{2J+1} \), or between \( x_{2J+1} \) and \( x_{2J+2} \). See Fig. 3 subplot (a).

We first consider the case \( 0 \leq \gamma < 1/2 \), for which the MTF-2 reads

\[ u_{0}^{n+1} = 2[(1 - \gamma)u_J^n + \gamma u_{J+1}^n] - [(1 - 2\gamma)u_{2J}^{n-1} + 2\gamma u_{2J+1}^{n-1}]. \]  

(17)

By Taylor expansion, we find the following modified equation.

\[ u_{0}^{n+1} - 2[(1 - \gamma)u_J^n + \gamma u_{J+1}^n] + [(1 - 2\gamma)u_{2J}^{n-1} + 2\gamma u_{2J+1}^{n-1}] = -\left( \frac{\partial}{\partial t} - J\eta \frac{\partial}{\partial x} \right) [\frac{\partial}{\partial t} - (J + 2\gamma)\eta \frac{\partial}{\partial x}] u(\Delta t)^2 + O((\Delta x)^3). \]  

(18)

We reformulate this to obtain a continuous form.

\[ \left( \frac{\partial}{\partial t} - J\eta \frac{\partial}{\partial x} \right) \left[ \frac{\partial}{\partial t} - (J + 2\gamma)\eta \frac{\partial}{\partial x} \right] u = O(\Delta x). \]  

(19)

To let the normal incidence waves propagate transparently, we may take either \( J\eta = c \) or \((J + 2\gamma)\eta = c \). Because the first choice requires \( c \) to be a multiple
of $\eta = \Delta x/\Delta t$, which is not satisfied in general, we recommend the second choice $J + 2\gamma = \frac{c}{\eta}$. This leads to $J = \left[\frac{c}{\eta}\right]$, and $a^* = \frac{1}{2}(c + \left[\frac{c}{\eta}\right] \eta)$.

In particular, considering $c < \eta$, we find $a^* = c/2$ and the corresponding reflection coefficient

$$R(\theta) = -\frac{1 - \cos \theta}{1 + \cos \theta}.$$  \hspace{1cm} (20)

Next, for the case $1/2 \leq \gamma < 1$, the MTF-2 takes the following form.

$$u_0^{n+1} = 2[(1 - \gamma)u_0^n + \gamma u_{J+1}^n] - [(2 - 2\gamma)u_{2J+1}^{n-1} + (2\gamma - 1)u_{2J+2}^{n-1}]. \hspace{1cm} (21)$$

We find its continuous form in the same way as the previous case.

$$[\frac{\partial}{\partial t} - (J + 1)\eta \frac{\partial}{\partial x}][\frac{\partial}{\partial t} - (J + 2\gamma - 1)\eta \frac{\partial}{\partial x}]u = O(\Delta x). \hspace{1cm} (22)$$

In this case, we may again show that $J = \left[\frac{c}{\eta}\right]$, and suggest a choice $a^* = \frac{1}{2}(c + \left[\frac{c}{\eta}\right] + 1)\eta)$. Due to the stability requirement, we actually have $J = 0$ and $a^* = (c + \eta)/2$. The reflection coefficient then reads

$$R(\theta) = -c - \eta \cos \theta \cdot \frac{1 - \cos \theta}{c + \eta \cos \theta} \cdot \frac{1 - \cos \theta}{1 + \cos \theta}. \hspace{1cm} (23)$$

For both cases, our choices differ from Liao’s suggestion of $a = c$. In comparison, the first boundary condition only absorbs the normal incidence wave, whereas the second boundary condition transparently absorbs also an incident wave at $\theta = \arccos(c/\eta)$. Therefore, we suggest for the MTF-2 boundary condition,

$$\frac{1}{2} \leq \gamma < 1, \quad a = a^{**} = (c + \eta)/2. \hspace{1cm} (24)$$

### 3.4 MTF-3

In Fig. 3 subplot (b), we depict the setting of numerical grids. There are three cases for MTF-3, depending on the value of $\gamma$. 

Figure 3: Grid points: (a) MTF-2; (b) MTF-3.
First, if \(0 \leq \gamma \leq 1/3\), MTF-3 may be put as
\[
up_{n+1} = 3[(1 - \gamma)u^n_0 + \gamma u^n_{J+1}] - 3[(1 - 2\gamma)u^{n-1}_{2J} + 2\gamma u^{n-1}_{2J+1}] + [(1 - 3\gamma)u^{n-2}_{3J} + 3\gamma u^{n-2}_{3J+1}].
\] (25)

We take Taylor expansions and find the continuous form
\[
(\frac{\partial}{\partial t} - J\eta\frac{\partial}{\partial x})^2[\frac{\partial}{\partial t} - (J + 3\gamma)\eta\frac{\partial}{\partial x}]u = O(\Delta x).
\] (26)

This suggests a choice of \(a = \left(2\left[\frac{c}{\eta}\right] + 2\eta + c\right)/3\). The corresponding reflection coefficient for \(J = 0\) is again given by (20).

In the mean time, if \(2/3 \leq \gamma < 1\), MTF-3 reads
\[
up_{n+1} = 3[(1 - \gamma)u^n_0 + \gamma u^n_{J+1}] - 3[(1 - 2\gamma)u^{n-1}_{2J} + (2\gamma - 1)u^{n-1}_{2J+1}] + [(3 - 3\gamma)u^{n-2}_{3J+2} + (3\gamma - 2)u^{n-2}_{3J+3}].
\] (27)

After some calculations, we find the continuous form
\[
(\frac{\partial}{\partial t} - (J + 1)\eta\frac{\partial}{\partial x})^2[\frac{\partial}{\partial t} - (J + 3\gamma - 2)\eta\frac{\partial}{\partial x}]u = O(\Delta x).
\] (28)

When we take an artificial wave propagation speed \(a = \left(2\left[\frac{c}{\eta}\right] + 2\eta + c\right)/3\), the reflection coefficient is
\[
R(\theta) = -\left(\frac{c - \eta \cos \theta}{c + \eta \cos \theta}\right)^2 \frac{1 - \cos \theta}{1 + \cos \theta}.
\] (29)

Finally, if \(1/3 \leq \gamma < 1/2\), MTF-3 reads
\[
up_{n+1} = 3[(1 - \gamma)u^n_0 + \gamma u^n_{J+1}] - 3[(1 - 2\gamma)u^{n-1}_{2J} + 2\gamma u^{n-1}_{2J+1}] + [(2 - 3\gamma)u^{n-2}_{3J+1} + (3\gamma - 1)u^{n-2}_{3J+2}].
\] (30)

It does not reach a favorable absorption. In fact, Taylor expansion gives
\[
u^n_{J+1} - 3[(1 - \gamma)u^n_0 + \gamma u^n_{J+1}] + 3[(1 - 2\gamma)u^{n-1}_{2J} + 2\gamma u^{n-1}_{2J+1}]
= [(2 - 3\gamma)u^{n-2}_{3J+1} + (3\gamma - 1)u^{n-2}_{3J+2}]
= (1 - 3\gamma)(\Delta x)^2u_{xx} + O(\Delta x^3).
\] (31)

It is similar for the case when \(1/2 \leq \gamma < 2/3\). As \(u_{xx} = 0\) is not a favorable absorbing boundary condition, it is not advisable to choose \(a\) such that the straight line with slope \(-a\) locates in the shadowed region of Fig. 3 subplot (b).

In summary, we suggest for MTF-3 the following choice.
\[
\frac{2}{3} \leq \gamma < 1, \quad a = a^{***} = (2\eta + c)/3.
\] (32)

We make some remarks. First, we suggest to take \(1 - 1/N \leq \gamma < 1\) and \(a = a^{(N)} = (\eta + c)/N\) for a general MTF-N boundary condition.
Secondly, high order approximations are preferable to maintain the accuracy of MTF in practical computations [21]. Compared with Higdon’s discrete absorbing boundary conditions, except the space-time extrapolation, it is easier to formulate the high order MTF boundary conditions which involve much less grid points. MTF-N is devised once we choose the artificial wave propagation speed \( a \) and the mesh ratio \( \eta \).

Thirdly, Liao suggested to take \( a = c \). Because \( \sum_{j=1}^{N} a_j/N = a \) for MTF-2 and MTF-3, the choice \( a = c \) does not transparently absorb the normal incident waves unless \( c/\eta \) is an integer.

### 3.5 Reflection coefficient of MTF-N with CTCS

In real computations, an absorbing boundary condition is adopted together with an interior scheme. Taking a uniform mesh with \( \Delta x = \Delta y \) and \( \eta = \Delta x/\Delta t \), we adopt the standard CTCS scheme for the wave equation (1).

\[
\eta^2(u_{j+1}^{n+1} - 2u_{j,m}^{n} + u_{j-1}^{n-1}) = c^2(u_{j+1,m}^{n} + u_{j-1,m}^{n} + u_{j,m+1}^{n} - 4u_{j,m}^{n}). \tag{33}
\]

Due to stability concerns, we restrict the mesh ratio \( \eta > \sqrt{2} \).

For a plane wave with wave number \( k \), frequency \( \omega \), and incidence angle \( \theta \), we define \( \xi = k\Delta x \), \( \xi_x = \xi \cos \theta \) and \( \xi_y = \xi \sin \theta \). By standard reflection coefficient analysis on an left artificial boundary \( x = 0 \), we obtain for MTF-N condition, that

\[
R(\xi, \theta) = \frac{1 - \sum_{\beta=1}^{N} (-1)^{\beta+1} C_{N}^{\beta} \left[ (1 - \gamma_{\beta}) + \gamma_{\beta} e^{i\xi_x} \right] e^{i(J_{\beta} \xi_x - \beta \omega \Delta t)}}{1 - \sum_{\beta=1}^{N} (-1)^{\beta+1} C_{N}^{\beta} \left[ (1 - \gamma_{\beta}) + \gamma_{\beta} e^{-i\xi_x} \right] e^{-i(J_{\beta} \xi_x + \beta \omega \Delta t)}}. \tag{34}
\]

Here the dispersion relation reads

\[
\eta^2 \sin^2 \frac{\omega \Delta t}{2} = c^2 \left( \sin^2 \frac{\xi_x}{2} + \sin^2 \frac{\xi_y}{2} \right). \tag{35}
\]

Similar to the situation for continuous forms, the reflection coefficient is a multiplication of those induced by single first order boundary conditions. For instance, when \( 0 \leq \gamma < 1/2 \), the reflection coefficient for MTF-2 (17) is

\[
R = \frac{1 - e^{i(J \xi_x - \omega \Delta t)}}{1 - e^{-i(J \xi_x + \omega \Delta t)}} \cdot \frac{1 - [(1 - 2\gamma) + 2\gamma e^{i\xi_x}] e^{i(J \xi_x - \omega \Delta t)}}{1 - [(1 - 2\gamma) + 2\gamma e^{-i\xi_x}] e^{-i(J \xi_x + \omega \Delta t)}}. \tag{36}
\]

To verify the above analysis, we plot the reflection coefficients in Figure 4, with \( c = 1 \), \( \eta = 10/7 \) and \( \xi = 0.5 \).

The reflection coefficients for MTF-1 in subplot (a) clearly show the influence of \( c = 1 \). Liao’s suggestion of \( a = c = 1 \) leads to a relatively better transmission, particularly for small incidence angles. Because \( \xi = 0.5 \) corresponds to a relatively large wave number, the long wave limit is not valid.
Figure 4: Reflection coefficient versus incidence angle: (a) MTF-1 (top-left); (b) MTF-2 (top-right); (c) MTF-3 (bottom-left); (d) MTF with linear and quadratic interpolations (bottom right).
and the normal incidence wave is not transparently absorbed. On the other hand, with Higdon’s space-time extrapolation choice $a = a_H = \eta$, a complete absorption is reached at $\theta = 45^\circ$. With other choices $a = 2$ and $a = 5$, full absorption is reached at $\theta = 60^\circ$ and $\theta = 78^\circ$, respectively. While the reflection coefficients are not so small over a wide range of incidence angles, MTF-1 is not good enough for applications.

In subplots (b) and (c), we depict the reflection coefficients for MTF-2 and MTF-3. The improvement over MTF-1 is evident. For MTF-2, the suggested value of $a^{**} = (\eta + c)/2 \approx 1.2143$ provides satisfactory absorption for incidence angles smaller than $45^\circ$. Higdon’s space-time extrapolation choice yields slightly better results for big incidence angles. Furthermore, a too big value of artificial wave propagation speed, e.g. $a = 5$, causes strong reflection. MTF-3 has a similar performance. We point out that at $a = 2$, strong reflections occur as the corresponding $\gamma = \{2/\eta\} = 0.4$. This gives rise to a situation in the shadowed region of Fig. 3.

In subplot (d), we observe that the upwind quadratic interpolation is effective to reduce the reflection for MTF-3 with a bad $a$. This agrees with Liao’s suggestion of using MTF-2Q or MTF-3Q in practical computations[21, 27]. In contrast, the improvement is less significant for MTF-1.

4 Numerical tests

In this section, we present some numerical tests to further elaborate the previous analysis. In particular, we perform computations for time harmonic waves, as well as a Gaussian initial data. All the computations are performed with a wave speed $c = 1$, mesh sizes $\Delta x = \Delta y = 0.025$, $\Delta t = 0.0175$, and therefore a mesh ratio $\eta = 10/7$.

4.1 Time harmonic waves

First, we compute with time harmonic waves to validate the reflection coefficient as a proper description for reflection property of MTF boundary conditions.
Figure 6: Evolution of the $L^2$-norm of the error in $\Omega_W$ with various incidence angles: (a) $a = 1$; (b) $a = a_H \approx 1.4286$.

Following the standard settings in [12, 13], we take the computing domain in Fig. 5 subplot (a). Numerical computations are performed with the CTCS scheme over $\Omega_C = [0, H] \times [-H, H]$, and the MTF boundary conditions on the left boundary $x = 0$. Numerical errors are calculated in a subdomain $\Omega_W = [0, 0.5] \times [-0.5, 0.5]$. To make comparison, we compute over $\Omega_B = [-H, H] \times [-H, H]$ and regard this result as an exact solution when it is confined on $\Omega_W$. Here $H$ is big enough to ensure that the boundaries of $\Omega_B$ and $\Omega_C$ do not influence the estimates in the subdomain $\Omega_W$. Noticing that a wave propagates at $c = 1$, we use $H > T + 0.5$ with $T$ the terminal time of the numerical simulation.

We compute for $\xi = k\Delta x = 0.5$, $\omega \Delta t = 2 \arcsin \left( \frac{1}{\eta} \sqrt{\sin^2 \frac{\xi \cos \theta}{2} + \sin^2 \frac{\xi \sin \theta}{2}} \right)$, and initial conditions for the first two time steps as follows.

\[
\begin{align*}
\{ u_{j,m}^0 & = e^{i(j \xi \cos \theta + m \xi \sin \theta)} , \\
       u_{j,m}^1 & = e^{i(j \xi \cos \theta + m \xi \sin \theta + \omega \Delta t)} .
\end{align*}
\]

(37)

Numerical results with MTF-1 are displayed in Fig. 6. Subplots (a) and (b) show the $L^2$-norm of the errors over $\Omega_W$ for the choices $a = 1$ and $a = a_H = \eta \approx 1.4286$, respectively.

In good agreement with the reflection coefficient discussion in Fig. 4 subplot (a), we observe that $a = 1$ absorbs better the reflection at the angle $30^\circ$ than those at $45^\circ$ or $60^\circ$. For $a = a_H$, the incident wave at $45^\circ$ is almost completely absorbed.

Furthermore, we show the numerical solutions at a particular point $(\Delta x, 0)$ near the boundary in Fig. 7. Similar to the $L^2$-norm, the time series settle down to a steady oscillation after an initial period. The oscillation amplitudes are consistent to the values of $R$. For instance, theoretically computed reflection coefficients are 0.03883027, 0.07683873, 0.17418255, 0.33577217 for incidence angles $0^\circ$, $30^\circ$, $45^\circ$ and $60^\circ$, respectively. In the numerical tests, the oscillation amplitudes are 0.039, 0.077, 0.175 and 0.335, respectively.

A peculiar phenomenon occurs for the normal incidence wave $\theta = 0^\circ$, which has a relatively stronger reflection than that depicted in Fig. 4. More detailed analysis
shows that there appears a certain amount of shift in the average value of the reflected error besides the oscillation, which leads to a much larger reflection than $|R|$. Apart from this, the oscillation amplitude agrees well with $|R|$.

From the above numerical results, we conclude that the reflection coefficient $R$ is a good description of the reflection for MTF boundary conditions, yet the normal incidence waves need a more careful study.

Next, we present the numerical results with MTF-2 at $a = a^{**} = (1 + \eta)/2 \approx 1.2143$. The error is shown in Fig. 8. The results are similar to the MTF-1, except for an even longer transient stage and larger transient oscillations. After this stage, the absorption effects are much better than MTF-1.

4.2 MTF-2 boundary condition with Gaussian initial data

Over the computing domain $\Omega_C = [-2, 2] \times [-2, 2]$ shown in Fig. 5 subplot (b), we adopt the CTCS scheme and the MTF-2 boundary condition to simulate wave propagations. The exact solution is obtained by computations over a larger domain $\Omega_B = [-H, H] \times [-H, H]$ with $H \geq 2 + T$.

We take a Gaussian source as the initial condition.

$$u(x, y, 0) = \begin{cases} e^{-30(x^2+y^2)}, & x^2 + y^2 < 0.25, \\ 0, & x^2 + y^2 \geq 0.25 \end{cases}, \quad u_t(x, y, 0) = 0. \quad (38)$$

It contains plane wave components over a range of both wavelengths and incidence angles. We compare the ratio of the $L^2$-norm of the numerical error to the $L^2$-norm of the initial data on $\Omega_C$. The re-scaled $L^2$-norms with $a = 1$, $a = a_H = \eta = 10/7 \approx 1.4286$, and $a = a^{**} = (1 + \eta)/2 \approx 1.2143$ are displayed in Fig. 9. It is evident that the suggested choice $a = a^{**}$ performs significantly better than other choices.

The numerical error evolution has different features at several stages. First, for $t \in (0, t_0)$ with $t_0 = H - 0.25 = 1.75$, the Gaussian source does not affect
Figure 8: The error at $(\Delta x,0)$ with different incidence angles, using the MTF-2 boundary condition with $a = a^\ast$.

Figure 9: The $L_2$-norm of the error in $\Omega_C$ with Gaussian source.
the boundary at this stage and no reflection appears. Reflection occurs during a second stage \( t \in [t_0, t_1] \) with \( t_1 \approx 2.58 \). This starts when the wavefronts reach at the middle of the artificial boundaries at around \( t_0 = 1.75 \) with a normal incidence angle, and further develops until the wavefronts reach at the corner of \( \Omega_C \) at \( t_1 = 2\sqrt{2} - 0.25 \approx 2.58 \). The error increases gradually during this period. Next, for \( t \in [t_1, t_2] \) with \( t_2 \approx t_0 + 2H \approx 5.75 \), wave fronts are absorbed along the boundaries and the reflected waves propagate toward the opposite boundaries. After \( t_2 \), this process of absorption, reflection and transmission repeats. Due to the mixture of propagation and reflection, the peak-valley feature becomes less distinct.

Furthermore, in Fig. 10 we plot the solutions near the center, at the points \((\Delta x, 0)\) and \((\Delta x, 2 - \Delta y)\). The most effective absorption is obtained with \( a = a^{**} \). In comparison, computations for \( a = 1 \) and \( a_H \) show reflections at around \( t \approx 2 \), as well as at a later stage, especially at around \( t \approx 6 \).

The numerical results demonstrate the effectiveness of the MTF-2 boundary condition with the suggested choice \( a = a^{**} \).

5 Conclusions

In this paper, we formulate and systematically analyze the MTF boundary conditions. The MTF conditions bear a nice feature of simplicity, which is particularly interesting for practical applications and for implementation of high order algorithms. The only parameters in such conditions are the artificial wave propagation speed and the space-time mesh ratio. By a study on the reflection coefficient for monochromatic waves with different incidence angles, we propose new and effective choices of the artificial wave propagation speed for various versions of MTF boundary conditions. We find that the reflection property relies heavily on the choice of the artificial wave propagation speed. The advantages of the proposed choices are demonstrated through numerical and theoretical calculations when a centered-time-centered-space scheme is adopted as the interior solver.

Moreover, we observe that when linear interpolation is used, MTF-3 does not effectively reduce the reflections if the mesh ratio falls into a 'bad' region. Reflection coefficient analysis shows that this may be changed if an upwind quadratic interpolation is used instead of the linear one.

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Figure 10: Evolution at $(\Delta x, 0)$ and $(\Delta x, 2 - \Delta y)$: (a) $a = 1$; (b) $a = a^{**} \approx 1.2143$; (c) $a = a_H \approx 1.4286$. 
References


